ON THE THEORY OF PLANE GAS FLOWS

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PMM Vol.23, No.1, 1959, pp. 201-208

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(Received 23 January 1958)

Starting from a certain simple tractable solution of a system of equations of the Chaplygin type for the plane motion of a gas, a method is presented by which other systems of equations of a type containing arbitrary constants in their coefficients can be obtained. By selecting the constants it is possible to obtain good approximations to the equations of adiabatic gas flow over a wide range of velocity variation. Peres [1,2] has proposed a similar method.

However, with the transformations employed in [1] and [2], important properties of the initial solutions are not preserved. In our work, after the application of each Legendre transformation and the generalization of the functions which generate the coefficients of the system of equations, the inverse transformation with these generalized functions is applied. As a result, such important properties of the initial flow as continuity of the subsonic flow into the supersonic domain and uniformity of the flow at infinity are preserved. The method is applied to gas flows with transition through sonic velocity. The Tricomi equation is taken for the initial equation. A better approximation to real flows is obtained over the range of relative velocity variation $0.1 < \lambda < 1.2$. The calculation of a family of nozzles is given.

1. Presentation of the method. From the condition of total differentials of the expressions

$$\cos\vartheta P_1(\lambda)\,d\varphi_1 - \sin\vartheta Q_1(\lambda)\,d\psi_1 = dx_1 \qquad (1.1)$$
$$\sin\vartheta P_1(\lambda)\,d\varphi_1 + \cos\vartheta Q_1(\lambda)\,d\psi_1 = dy_1$$

where $P_1(\lambda)$ and $Q_1(\lambda)$ are certain given functions of the independent variable λ, the following system of equations for the unknown functions $\phi_1(\theta, \lambda)$ and $\psi_1(\theta, \lambda)$ can be derived:

$$\frac{\partial\varphi_1}{\partial\vartheta} = -\frac{Q_1(\lambda)}{P_1'(\lambda)}\frac{\partial\psi_1}{\partial\lambda}, \qquad \frac{\partial\varphi_1}{\partial\lambda} = \frac{Q_1'(\lambda)}{P_1(\lambda)}\frac{\partial\psi_1}{\partial\vartheta} \qquad (1.2)$$

In canonical form the system (1.2) has the form:

$$\frac{\partial \varphi_1}{\partial \vartheta} = \pm \sqrt{K_1} \frac{\partial \psi_1}{\partial \sigma}, \qquad \frac{\partial \varphi_1}{\partial \sigma} = \mp \sqrt{K_1} \frac{\partial \psi_1}{\partial \vartheta} \tag{1.3}$$

where

$$\sqrt{K_1} = \left(\frac{Q_1(\lambda) Q_1'(\lambda)}{P_1(\lambda) P_1'(\lambda)}\right)^{1/2} \qquad \sigma(\lambda) = \int_1^\lambda \left(\frac{P_1'(\lambda) Q_1'(\lambda)}{P_1(\lambda) Q_1(\lambda)}\right)^{1/2} d\lambda \tag{1.4}$$

Formulas (1.4) can be transformed into the form

$$\frac{dQ_1}{d\sigma} = \mp \sqrt{K_1}\, P_1, \qquad Q_1 = \mp \sqrt{K_1} \frac{dP_1}{d\sigma} \tag{1.5}$$

According to the behavior of functions $P_1(\lambda)$ and $Q_1(\lambda)$ and their derivatives, system (1.3) and formulas (1.5) are taken together with either the upper or lower signs in front of $\sqrt{K_1}$.

In particular, with

$$Q_1(\lambda) = Q(\lambda) = \lambda^{-1}\left(1 - \frac{\lambda^2}{h^2}\right)^{-\frac{1}{\varkappa - 1}} \tag{1.6}$$

the system (1.3) is the Chaplygin system of equations for the plane motion of a gas. For this case the functions $\phi_1 = \phi$ and $\psi_1 = \psi$ will be the velocity potential and the stream function, $x_1 = x$ and $y_1 = y$ the Cartesian coordinates of the plane of flow, θ the angle at which the velocity vector is inclined to the x-axis, λ the magnitude of the relative velocity, and $h^2 = (\kappa + 1)/(\kappa - 1)$. With (1.6)

$$\sqrt{K_1} = \sqrt{K} = \left(\frac{1 - \lambda^2}{(1 - \lambda^2 / h^2) h^2}\right)^{1/2}, \qquad \sigma(\lambda) = s(\lambda) = \int \left(\frac{1 - \lambda^2}{1 - \lambda^2 / h^2}\right)^{1/2} \frac{d\lambda}{\lambda} \tag{1.7}$$

and the upper signs must be placed in front of $\sqrt{K}$ in formulas (1.3) and (1.5). The canonical form of the Chaplygin equations is convenient for investigation and was first widely used in the work of Khristianovich [4,5,6]. With a given function $\sqrt{K_1(\sigma)}$ system (1.3) can be obtained with various functions P_1 and Q_1. In fact, with a given function $\sqrt{K_1}$ formulas (1.5) represent a system of equations with respect to P_1 and Q_1. Eliminating one of the unknown functions, we obtain a linear differential equation of second order with respect to the other unknown function. Therefore, in general form functions P_1 and Q_1 will depend on two arbitrary constants which do not enter into the expression for $\sqrt{K(\sigma)}$.

We will now present a way of obtaining from the system of equations (1.3) analogous equations with new arbitrary constants contained in their coefficients. We pass from the functions ϕ_1, ψ_1 to the functions Φ, Ψ with the help of the Legendre transformations:

$$\Phi = x_1 \frac{\partial \varphi_1}{\partial x_1} + y_1 \frac{\partial \varphi_1}{\partial y_1} - \varphi_1, \qquad \Psi = x_1 \frac{\partial \psi_1}{\partial x_1} + y_1 \frac{\partial \psi_1}{\partial y_1} - \psi_1 \tag{1.8}$$

We have

$$x_1 = \Phi_{u_1} = -\Psi_{t_1}, \qquad y_1 = \Phi_{v_1} = \Psi_{r_1} \tag{1.9}$$

where

$$u_1 = P_1^{-1}(\lambda)\cos\vartheta, \quad v_1 = P_1^{-1}(\lambda)\sin\vartheta, \quad r_1 = Q_1^{-1}(\lambda)\cos\vartheta, \quad t_1 = Q_1^{-1}(\lambda)\sin\vartheta \tag{1.10}$$

If in system (1.9) we pass from the variables u_1, v_1, r_1, t_1 to the independent variables θ, λ and then further reduce to canonical form, we finally obtain

$$\frac{\partial \Psi}{\partial \vartheta} = \mp \chi_1(\lambda)\frac{\partial \Phi}{\partial \sigma}, \qquad \frac{\partial \Psi}{\partial \sigma} = \pm \chi_1(\lambda)\frac{\partial \Phi}{\partial \vartheta} \left(\chi_1(\lambda) = \sqrt{K_1}\frac{P_1^2}{Q_1^2}\right) \tag{1.11}$$

The functions $P_1(\lambda)$, $Q_1(\lambda)$ are a particular solution of the system of equations

$$\frac{P_2^{*2}}{Q_2^{*2}}\left(\frac{Q_2^*(\lambda)\,Q_2^{*\prime}(\lambda)}{P_2^*(\lambda)\,P_2^{*\prime}(\lambda)}\right)^{1/2} = \chi_1(\lambda), \qquad \frac{P_2^{*\prime}(\lambda)\,Q_2^{*\prime}(\lambda)}{P_2^*(\lambda)Q_2^*(\lambda)} = \sigma'^2(\lambda) \tag{1.12}$$

where $P_2{}^*$ and $Q_2{}^*$ are the unknown functions. The system (1.12) is transformed to the form

$$\frac{dq_2^*}{d\sigma} = \pm \chi_1 p_2^*, \qquad q_2^* = \pm \chi_1 \frac{dp_2^*}{d\sigma} \qquad (p_2^* = P_2^{*-1},\ q_2^* = Q_2^{*-1}) \tag{1.13}$$

Hence

$$\frac{d^2 p_2^*}{d\sigma^2} + \frac{d\ln\chi_1}{d\sigma}\frac{dp_2^*}{d\sigma} - p_2^* = 0 \tag{1.14}$$

Employing the Liouville formula to calculate the general solution of equation (1.14) and taking (1.13) into account, we obtain

$$p_2^* = p_1(1 + a_1 J_1), \quad J_1 = \int_0^\sigma \frac{Q_1^2}{\sqrt{K_1}}\,d\sigma$$

$$q_2^* = q_1(1 \pm a_1 P_{10}Q_{10} - a_1 J_2), \quad J_2 = \int_0^\sigma \sqrt{K_1}\,P_1^2\,d\sigma \tag{1.15}$$

where P_{10} and Q_{10} are the values of P_1 and Q_1 with $\sigma = 0 (\lambda = 1)$, and a_1 is a constant of integration. The functions $p_2{}^*$ and $q_2{}^*$ are calculated to within an arbitrary constant factor which does not affect the generality of the investigation. By an inverse transformation from the functions Φ, Ψ to the functions ϕ_2, ψ_2 according to the formulas

$$\varphi_2 = u_2^* \frac{\partial \Phi}{\partial u_2^*} + v_2^* \frac{\partial \Phi}{\partial v_2^*} - \Phi, \qquad \psi_2 = r_2^* \frac{\partial \Psi}{\partial r_2^*} + t_2^* \frac{\partial \Psi}{\partial t_2^*} - \Psi \tag{1.16}$$

where

$$u_2^* = p_2^* (\lambda) \cos \vartheta, \quad v_2^* = p_2^* (\lambda) \sin \vartheta, \quad r_2^* = q_2^* (\lambda) \cos \vartheta, \quad t_2^* = q_2^* (\lambda) \sin \vartheta \tag{1.17}$$

we arrive at the system of equations

$$\frac{\partial \varphi_2}{\partial v} = \pm \sqrt{K_2} \frac{\partial \psi_2}{\partial \sigma}, \qquad \frac{\partial \varphi_2}{\partial \sigma} = \mp \sqrt{K_2} \frac{\partial \psi_2}{\partial \vartheta} \tag{1.18}$$

where

$$\sqrt{K_2} = \left(\frac{Q_2^* (\lambda) Q_2^{*\prime} (\lambda)}{P_2^* (\lambda) P_2^{*\prime} (\lambda)}\right)^{1/2} = \sqrt{K_1} \left(\frac{1 + a_1 J_1}{1 \pm a_1 P_{10} Q_{10} - a_1 J_2}\right)^2 \tag{1.19}$$

The function $\sqrt{K_2}$ may contain two more arbitrary constants than $\sqrt{K_1}$. The second essential constant c_1 is contained in functions P_1 and Q_1 if they are computed with the general solution of system (1.5) for a given function $\sqrt{K_1(\sigma)}$.

Increasing by one the indices in (1.1) and (1.5), we obtain formulas for computing the plane x_2, y_2 which corresponds to system (1.18), and we also obtain a system of equations for functions P_2 and Q_2, which are computed from their particular solutions $P_2 = P_2^*$, $Q_2 = Q_2^*$. To within an arbitrary constant factor we obtain

$$P_2 = P_2^* \left(1 + c_2 \int_0^\sigma \frac{q_2^{*2}}{\chi_1} d\sigma\right), \qquad Q_2 = Q_1 \left(1 \mp c_2 p_{20}^* q_{20}^* - c_2 \int_0^\sigma \chi_1 p_2^{*2} d\sigma\right) \tag{1.20}$$

where p_{20}^* and q_{20}^* are the values of p_2^* and q_2^* with $\sigma = 0$, and c_2 is a constant of integration. This method of acquiring constants can be continued farther. Increasing the indices in (1.19) by one, we obtain a formula for $\sqrt{K_3}$, etc. The function $\sqrt{K_3}$ will already contain four arbitrary constants more than $\sqrt{K_1}$. Supposing the initial system (1.3) sufficiently simple for solution, by selecting $2(n - 1)$ arbitrary constants we can try to make $\sqrt{K_n}$ approximate the $\sqrt{K}$ of adiabatic gas flow. The dependence between ϕ_n, ψ_n and ϕ_1, ψ_1 will be apparent if we find them for $n = 2$.

We will denote

$$\Phi_{u_2^*} = -\Psi_{t_2^*} = x_2^*, \qquad \Phi_{v_2^*} = \Psi_{r_2^*} = y_2^* \tag{1.21}$$

Taking into account formulas (1.16), (1.13), (1.10), (1.9) and (1.5), after simple calculations we obtain

$$x_2^* = \left(\frac{P_2^*}{P_1}\sin^2\vartheta + \frac{Q_2^*}{Q_1}\cos^2\vartheta\right)x_1 + \left(\frac{Q_2^*}{Q_1} - \frac{P_2^*}{P_1}\right)\sin\vartheta\cos\vartheta y_1$$

$$y_2^* = \left(\frac{Q_2^*}{Q_1} - \frac{P_2^*}{P_1}\right)\sin\vartheta\cos\vartheta x_1 + \left(\frac{P_2^*}{P_1}\cos^2\vartheta + \frac{Q_2^*}{Q_1}\sin^2\vartheta\right)y_1 \qquad (1.22)$$

From formulas (1.8), (1.16), (1.21) and (1.22) it follows that

$$\varphi_2 = \varphi_1 + \left(\frac{Q_2^*}{Q_1P_2^*} - \frac{1}{P_1}\right)(\cos\vartheta x_1 + \sin\vartheta y_1)$$

$$\psi_2 = \psi_1 + \left(\frac{P_2^*}{P_1Q_2^*} - \frac{1}{Q_1}\right)(\cos\vartheta y_1 - \sin\vartheta x_1) \qquad (1.23)$$

Increasing by one the indices in (1.23), we obtain formulas for functions ϕ_3, ψ_3, etc. Functions P_3^*, Q_3^* are calculated according to formulas analogous to (1.15). On the basis of formulas (1.23) and (1.1) we conclude that function ψ_2 preserves a series of important properties of the initial flow. For instance, if ψ_1 has a singularity which represents an undisturbed translational flow at infinity, then ψ_2 also contains this singularity. At the transition line the condition of continuity of the subsonic flow into the supersonic domain is also preserved [5,7]. For $P_2^* \neq P_1$, $Q_2^* = Q_1$, system (1.18) coincides with system (1.3). On the basis of (1.23), $\psi_2 = \psi_1$ in this case.

We note that coinciding systems of equations are also obtained in analogous circumstances by Peres, but that every concrete solution ψ_1 varies according to the formula $\psi_2 = \psi_1 + \partial^2\psi_1/\partial\theta^2$. Consequently, the transformations used in [1,2] do not preserve such important properties of the initial flow as, for instance, continuity of the subsonic flow into the supersonic domain [7].

2. Application of the method. Calculation of nozzles. We will apply the method to gas flows with transition through sonic velocity. In the initial system (1.3) we assume

$$\sigma(\lambda) = s(\lambda), \qquad \sqrt{K_1} = As^{1/3} = -\left(\frac{2}{3}\right)^{1/3} A\eta^{1/2} \quad (A < 0) \qquad (2.1)$$

The variable $\eta = (-3/2\ S)^{2/3}$ takes positive values in the elliptic region and negative values in the hyperbolic region. With (2.1) we obtain for P_1 and Q_1 the Airy equation

$$\frac{d^2P_1(\eta)}{d\eta^2} - \eta P_1(\eta) = 0 \qquad (2.2)$$

We have

$$P_1 = c_1k(\eta) + c_2l(\eta), \qquad Q_1 = \mp\left(\frac{2}{3}\right)^{1/3} A\,(c_1k'(\eta) + c_2l'(\eta)) \qquad (2.3)$$

where $k(\eta)$ and $l(\eta)$ are linearly independent solutions of equation (2.2), represented by the following series which are convergent for all values of η :

$$k(\eta) = 1.0899\left(1 + \frac{\eta^3}{2\cdot 3} + \frac{\eta^6}{(2\cdot 5)(3\cdot 6)} + \ldots\right) + 0.7946\left(\eta + \frac{\eta^4}{3\cdot 4} + \frac{\eta^7}{(3\cdot 6)(4\cdot 7)} + \ldots\right) \quad (2.4)$$

$$l(\eta) = 0.6293\left(1 + \frac{\eta^3}{2\cdot 3} + \frac{\eta^6}{(2\cdot 5)(3\cdot 6)} + \ldots\right) - 0.4587\left(\eta + \frac{\eta^4}{3\cdot 4} + \frac{\eta^7}{(3\cdot 6)(4\cdot 7)} + \ldots\right)$$

Tables of these functions have been computed by Fok [8]. With (2.1) the system (1.3) is the principal part of the Chaplygin system of equations in the neighborhood of $\lambda = 1$, if

$$A = A_0 = -3^{1/3}\left(\frac{\varkappa+1}{2}\right)^{\frac{\varkappa+2}{3(\varkappa-1)}}$$

and if the upper signs are taken in front of $\sqrt{K_1}$.

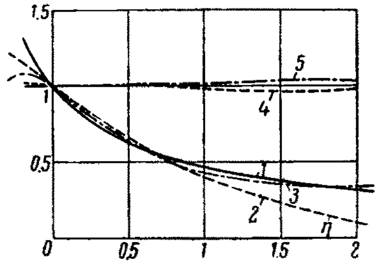

Fig. 1.

In Fig. 1 curve (1) depicts with $\kappa = 1.4$ the function

$$f = \frac{\sqrt{K}}{A_0 s^{1/3}} = -\left(\frac{3}{2}\right)^{1/3}\frac{\sqrt{K}}{A_0\eta^{1/2}} \quad (2.5)$$

whose deviation from unity is an indication of the site of the transonic range of variation of η in which solutions of the system (1.3) with (2.1) and $A = A_0$ can represent real flows. Curve (2) shows the dependence of λ on η. But in many problems such as, for instance, the calculation of nozzles we have larger intervals of velocity variation.

For a more precise approximation to $\sqrt{K}$ we shall here confine ourselves to the function $\sqrt{K_2}$. When the condition

$$A = A_0(1 \pm a_1 P_{10} Q_{10})^2 \quad (2.6)$$

is satisfied, the function

$$f_* = -\left(\frac{3}{2}\right)^{1/3}\frac{\sqrt{K_2}}{A_0\eta^{1/2}} \quad (2.7)$$

equals unity at the point $\eta = 0$. With values of $c_1 = 0$, $c_2 = 1$ and

$(2/3)^{1/3} a_1 A = -1.5$ curve (3) of function f_* is close to the exact curve over a large interval of variation of η (Fig. 1). From (2.6) we find that $A = -0.7773$. We now note that with $c_1 = 0$ the positive functions P_1 and Q_1 vary oppositely from the functions of the real flow. Therefore, in the case under consideration, all the formulas in Section 1 are taken with the lower signs. Thus, the functions P_2 and Q_2 satisfy a system of equations of type (1.5) with a plus sign in front of $\sqrt{K_2}$, whereas the functions P and Q of an adiabatic gas flow satisfy an analogous system of equations with a minus sign in front of $\sqrt{K}$. From the proximity of $\sqrt{K_2}$ to $\sqrt{K}$ it follows that the functions P_* and Q_* of the system of equations

$$\frac{dQ_*}{ds} = -\sqrt{K_2} P_*, \qquad Q_* = -\sqrt{K_2} \frac{dP_*}{ds} \tag{2.8}$$

can always be chosen close to the functions P and Q. For this it is sufficient to require that P_* and Q_* be coincident with the exact values for some λ within the interval where $\sqrt{K_2}$ is approximately equal to $\sqrt{K}$. In accordance with the particular solutions $P_2 = P_2{}^*$, $Q_2 = -Q_2{}^*$ we calculate the general solution

$$P_* = b_1 P_2{}^* \left(1 + b_2 \int_0^s \frac{q_2{}^{*2}}{\chi_1} ds\right), \qquad Q_* = -bQ_2{}^* \left(1 + b_2 p_{20}{}^* q_{20}{}^* - b_2 \int_0^s \chi_1 p_2{}^{*2} ds\right) \tag{2.9}$$

With values of the constants $b_1 = 1.589$ and $b_2 = -0.9702$, functions P_* and Q_* coincide with the exact values at the point $\lambda = 1$. In Fig. 1 curves (4) and (5) represent functions P/P_* and Q/Q_*. The system of equations of the form (1.18) with the upper signs in front of $\sqrt{K_2}$ corresponds to formulas (2.8). We have $\phi_* = -\phi_2$ and $\psi_* = \psi_2$, where ϕ_* and ψ_* are the velocity potential and stream function of the approximation to adiabatic flow achieved. Taking into account that in formulas of type (1.1) with index 2 for the case $P_2 = P_2{}^*$, $Q_2 = Q_2{}^*$ we have $x_2 = x_2{}^*$ and $y_2 = y_2{}^*$, for the calculation of the plane of the gas flow we obtain the following formulas:

$$dx_* = \left(\frac{Q_*}{Q_2{}^*}\sin^2\vartheta - \frac{P_*}{P_2{}^*}\cos^2\vartheta\right) dx_2{}^* - \sin\vartheta\cos\vartheta\left(\frac{P_*}{P_2{}^*} + \frac{Q_*}{Q_2{}^*}\right) dy_2{}^*$$

$$dy_* = -\left(\frac{P_*}{P_2{}^*} + \frac{Q_*}{Q_2{}^*}\right)\sin\vartheta\cos\vartheta\, dx_2{}^* + \left(\frac{Q_*}{Q_2{}^*}\cos^2\vartheta - \frac{P_*}{P_2{}^*}\sin^2\vartheta\right) dy_2{}^* \tag{2.10}$$

We consider the flow velocity λ. From the proximity of $\sqrt{K_2}$ to $\sqrt{K}$ it follows that the results are not essentially changed if the system of equations with respect to ϕ_* and ψ_* is considered to be exact for a fictitious gas and if, in accordance with this interpretation, the magnitude of the velocity is determined according to the formula $P_*^{-1}(\lambda)$.

For calculating nozzles from initial data we will take the following solutions of system (1.3) with (2.1), found by Fal'kovich:

$$\psi_1 = \alpha(\vartheta, \eta) + d_1\beta(\vartheta, \eta) \tag{2.11}$$

$$\alpha(\vartheta, \eta) = -(^3/_2)^{1/_3}\{(\vartheta + \sqrt{\vartheta^2 + {}^4/_9\eta^3})^{1/_3} + (\vartheta - \sqrt{\vartheta^2 + {}^4/_9\eta^3})^{1/_3}\} \tag{2.12}$$

$$\beta(\vartheta, \eta) = \left(\frac{2\eta_0^3}{\eta_0^3 + \eta^3 + {}^9/_4\vartheta^2}\right)^{1/_4} F\left(\frac{1}{12}, \frac{7}{12}, 1, 1 - \frac{4\eta_0^3\eta^3}{(\eta_0^3 + \eta^3 + {}^9/_4\vartheta^2)^2}\right) \times$$
$$\times \operatorname{arc\,tg} \frac{6\eta_0^{3/_2}\vartheta}{\eta^3 - \eta_0^3 + {}^9/_4\vartheta^2} \tag{2.13}$$

where $F(a, b, c, z)$ is the hypergeometric function and d_1 is an arbitrary constant [9,10]. Solution (2.11) realizes a family of nozzles whose upstream flow due to the function $\beta(\sigma, \eta)$ tends to the uniform subsonic velocity with corresponding magnitude η_0. With $\eta < 0$ the argument of the function F is larger than unity. According to the formula for the analytic continuation of the hypergeometric series [11], it follows that for $\eta < 0$

$$\beta(\vartheta, \eta) = \left(\frac{2\eta_0^3}{\eta^3 + \eta_0^3 + {}^9/_4\vartheta^2}\right)^{1/_4} \left\{\frac{\Gamma(^1/_3)}{\Gamma(^{11}/_{12})\Gamma(^5/_{12})} F\left(\frac{1}{12}, \frac{7}{12}, \frac{2}{3}, \frac{4\eta_0^3\eta^3}{(\eta_0^3 + \eta^3 + {}^9/_4\vartheta^2)}\right) + \right.$$
$$\left. + \frac{2^{2/_3}\Gamma(-^1/_3)\eta_0\eta}{\Gamma(^1/_{12})\Gamma(^7/_{12})}\left(\eta^3 + \eta_0^3 + \frac{9}{4}\vartheta^2\right)^{-2/_3} F\left(\frac{5}{12}, \frac{11}{12}, \frac{4}{3}; \frac{4\eta_0^3\eta^3}{(\eta^3 + \eta_0^3 + {}^9/_4\vartheta^2)^2}\right)\right\} \times$$
$$\times \operatorname{arc\,tg}\left(\frac{6\eta_0^{3/_2}\vartheta}{\eta^3 - \eta_0^3 + {}^9/_4\vartheta^2}\right) \tag{2.14}$$

The function $a(\theta, \eta)$ in the neighborhood of $\eta = 0$ is the principal part of the solution (2.11) which guarantees that the continuity condition is satisfied [7].

By x_{*1}, y_{*1}, ψ_{*1} we denote the magnitudes of x_*, y_*, ψ_* which correspond to the initial solution (2.12). Let x_{*2}, y_{*2}, ψ_{*2} correspond to function (2.13). For greater diversity in the choice of nozzle forms we will add the solution

$$\psi_{*3} = \vartheta \tag{2.15}$$

We have

$$x_{*3} = Q_*(\eta)\cos\vartheta - Q_*(0), \qquad y_{*3} = Q_*(\eta)\sin\vartheta \tag{2.16}$$

Thus, the family of nozzles

$$\psi_* = \psi_{*1} + d_1\psi_{*2} + d_2\psi_{*3}, \quad x_* = x_{*1} + d_1x_{*2} + d_2x_{*3}, \quad y_* = y_{*1} + d_1y_{*2} + d_2y_{*3} \tag{2.17}$$

is calculated according to the above formulas and they depend on two arbitrary constants d_1 and d_2. We have computed these functions in the variables a, η. Solving (2.12) with respect to θ, we obtain

$$\vartheta = -\eta\alpha - \frac{\alpha^3}{3} \tag{2.18}$$

All the necessary integrations are carried out initially for η along the axis of symmetry ($\theta = 0$), and then along a line η = const. The point $\theta = 0$, $\eta = 0$ corresponds to the origin of the coordinates $x_{*i} = y_{*i} = 0$ (i = 1, 2, 3).

Functions ψ_{*i}, x_{*i}, y_{*i} $(i = 1, 2, 3)$ have been tabulated for a series of subsonic values of η in the interval of variation of α from zero to unity, and can be obtained from the author.

The coordinates of the nozzle wall $\psi_* = \text{const}$ are determined on each line $\eta = \text{const}$ by integrating with respect to α. Fig. 2 shows a nozzle with $d_1 = d_2 = 0$ and $\psi_{*1} = 0.388$.

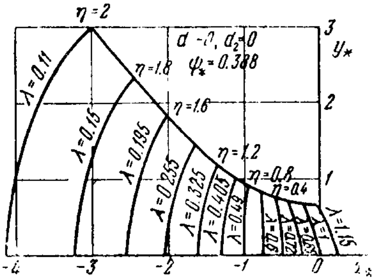

Fig. 2.

It is possible to use the given nozzle form to determine the outflow from a container of gas with transition through sonic velocity. The walls of the nozzle can be computed up to a smooth junction with the walls of the container.

For nozzles with subsonic translational flow at infinity the magnitude of the half-width on the basis of formulas (1.23) and (2.13) cannot be larger than

$$\frac{d_1}{2}\,\frac{p_1(\tau_0)}{p^*_2(\tau_0)}\,\frac{q^*_2(\tau_0)}{q_1(\tau_0)}\,\pi$$

Fig. 3 shows a nozzle with $d_1 = 1/2$, $d_2 = 0$ and $\eta_0 = 21$. Because of the effect of the function ψ_{*2}, the transonic section of the nozzle will

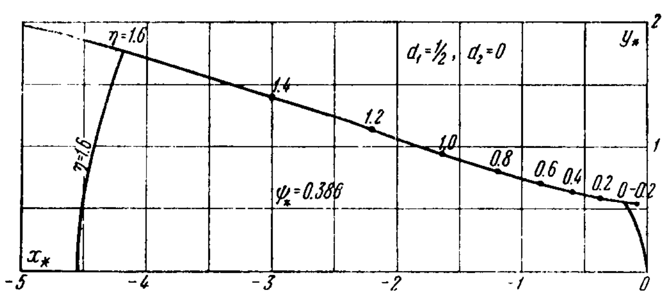

Fig. 3.

be less steep than that obtained with $d_1 = d_2 = 0$. By an appropriate choice of d_2 the effect of the second term $d_1\psi_{*2}$ can be cancelled in the neighborhood of $\eta = 0$.

According to [7] satisfaction of the continuity condition guarantees the potential nature of the supersonic flow only up to the characteristic which is tangent to the axis of symmetry of the transition (Fig. 4).

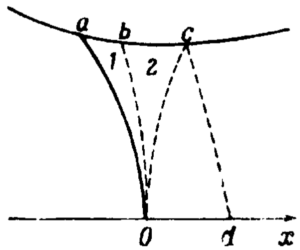

Fig. 4.

Since the calculation of the supersonic part of the nozzle is more conveniently carried out from the characteristic, and since nozzles with steep walls are of interest to us, we may expect that on the characteristic of the second family *cd* we are already sufficiently far from sonic velocity to use the well-known approximate solutions of the fundamental boundary value problems of supersonic gas flow [12, 13]. The flow in the region *ocd* will be potential if the Jacobian

$$J = \frac{D(\varphi_*, \psi_*)}{D(\vartheta, \lambda)} \neq 0$$

for all points of this region. In the independent variables θ, η proof of the potential nature of the flow is reduced to verification of the inequality

$$\frac{\partial \psi_*}{\partial \vartheta} \neq \pm \frac{1}{\sqrt{-\eta}} \frac{\partial \psi_*}{\partial \eta} \tag{2.19}$$

the validity of which can be demonstrated.

The transonic part of the nozzle can be calculated approximately by the formulas

$$\vartheta = \vartheta_0 + \left(\frac{d\vartheta}{\partial \eta}\right)_{\eta=0} (\eta - \eta_0) + \frac{1}{2}\left(\frac{d^2\vartheta}{d\eta^2}\right)_{\eta=0} (\eta - \eta_0)^2$$

$$y^* = y^*_0 + \left(\frac{dy^*}{dx^*}\right)_{\eta=0} (x^* - x^*_0) + \frac{1}{2}\left(\frac{d^2y^*}{dx^{*2}}\right)_{\eta=0} (x^* - x^*_0)^2 + \frac{1}{6}\left(\frac{d^3y^*}{dx^{*3}}\right)_{\eta=0} (x^* - x^*_0)^3 \tag{2.20}$$

where θ_0, x_0^*, y_0^* are the values of the quantities on the line $\eta = 0$ and the derivatives are taken along a streamline.

Along $\psi^* =$ const we have

$$\frac{d\vartheta}{d\eta} = -\frac{\partial\psi^*}{\partial\eta} \Big/ \frac{\partial\psi^*}{\partial\vartheta}, \qquad \frac{d^2\vartheta}{d\eta^2} = \frac{2\psi^*_\eta\psi^*_{\vartheta\eta}}{\psi^{*2}_\vartheta} - \frac{\psi^*_{\eta\eta}}{\psi^*_\vartheta} - \frac{\psi^{*2}_\eta\psi^*_{\vartheta\vartheta}}{\psi^{*3}_\vartheta} \tag{2.21}$$

With the help of formulas (2.3), (2.12), etc. it is not difficult to compute all the necessary quantities. For instance,

$$\left(\frac{dy^*}{dx^*}\right)_{\eta=0} = \operatorname{tg}\vartheta_0, \qquad \left(\frac{d^2y^*}{dx^{*2}}\right)_{\eta=0} = \frac{1.5^{1/3}}{\cos^3\vartheta_0(\psi^*_\eta)_{\eta=0}} \tag{2.22}$$

$$\left(\frac{\partial\psi^*}{\partial\eta}\right)_{\eta=0} = (1-a_1 P_1(0)Q_1(0))\left[\left(\frac{\partial\psi}{\partial\eta}\right)_{\eta=0} - \frac{1.5^{1/3}a_1 Q_1(0)}{A}(\cos\vartheta y_1 - \sin\vartheta x_1)_{\eta=0}\right]$$

$$\left(\frac{\partial\psi^*}{\partial\vartheta}\right)_{\eta=0} = (1-a_1 P_1(0)Q_1(0))\left(\frac{\partial\psi_1}{\partial\vartheta}\right)_{\eta=0} + a_1 P_1(0)(\sin\vartheta y_1 + \cos\vartheta x_1)_{\eta=0} \tag{2.23}$$

$$(\cos\vartheta y_1 - \sin\vartheta x_1)_{\eta=0} = -3^{1/3}Q_1(0)\,\vartheta_0^{1/3} + \left(d_2 - \frac{6\cdot 2^{1/6}}{\eta_0^{3/2}}\frac{\Gamma(1/3)}{\Gamma(11/12)\Gamma(9/12)}d_1\right)\vartheta_0 + O(\vartheta_0^{5/3})$$

$$(\sin\vartheta y_1 + \cos\vartheta x_1)_{\eta=0} = -\frac{3^{1/3}P_1(0)}{2^{2/3}}\vartheta_0^{2/3} - \frac{3^{4/3}Q_1(0)}{4}\vartheta_0^{4/3} + O(\vartheta_0^2) \tag{2.24}$$

$$\left(\frac{\partial\psi_1}{\partial\vartheta}\right)_{\eta=0} = -(3\vartheta_0)^{-2/3} - \frac{6\cdot 2^{1/6}}{\eta_0^{3/2}}\frac{\Gamma(1/3)d_1}{\Gamma(11/12)\Gamma(5/12)} + d_2 + O(\vartheta_0^2)$$

$$\left(\frac{\partial\psi_1}{\partial\eta}\right)_{\eta=0} = (3\vartheta_0)^{-1/3} - \frac{6.2^{5/6}\Gamma(-1/3)\,d_1}{\Gamma(1/12)\Gamma(7/12)\,\eta_0^{5/2}} + O(\vartheta_0^3) \tag{2.25}$$

Along the characteristic *oc* we have

$$\vartheta = \frac{2}{3}(-\eta)^{3/2} \tag{2.26}$$

Solving the system of equations (2.26) and (2.20), we find the values of $\theta = \theta_c$ and $\eta = \eta_c$ at point *c*. Along the characteristic *cd* we have

$$\vartheta + \frac{2}{3}(-\eta)^{3/2} = \vartheta_c + \frac{2}{3}(-\eta_c)^{3/2} \tag{2.27}$$

We compute the coordinates x_*, y_* along *cd* by formulas (2.10), (1.22) and (1.1). The values of x_1, y_1 which correspond to the function (2.12) in the variables η, α are equal to

$$x_1(\alpha,\eta) = x_1(0,\eta) - \left(\frac{2}{3}\right)^{1/3} AP_1(\eta)\int_0^\alpha \cos\left(\eta\alpha + \frac{\alpha^3}{3}\right)\alpha d\alpha + Q_1(\eta)\int_0^\alpha \sin\left(\eta\alpha + \frac{\alpha^3}{3}\right)d\alpha$$

$$y_1(\alpha,\eta) = \left(\frac{2}{3}\right)^{1/3} AP_1(\eta)\int_0^\alpha \sin\left(\eta\alpha + \frac{\alpha^3}{3}\right)\alpha d\alpha + Q_1(\eta)\int_0^\alpha \cos\left(\eta\alpha + \frac{\alpha^3}{3}\right)d\alpha \tag{2.28}$$

The solution (2.12) is not unique in the region *aodc*. For the regions *obc* and *ocd* we have respectively

$$\psi_1=\alpha_1=\left(\frac{3}{2}\right)^{1/3}(\xi+\nu)^{1/3}\left\{\cos\left(\frac{1}{3}\operatorname{arc\,tg}\frac{2\sqrt{\nu\xi}}{\nu-\xi}\right)+\sqrt{3}\sin\left(\frac{1}{3}\operatorname{arc\,tg}\frac{2\sqrt{\nu\xi}}{\nu-\xi}\right)\right.$$

$$\psi_1=\alpha_2=\left(\frac{3}{2}\right)^{1/3}(\xi+\nu)^{1/3}\left\{\cos\left(\frac{1}{3}\operatorname{arc\,tg}\frac{2\sqrt{\nu\xi}}{\nu-\xi}\right)-\sqrt{3}\sin\left(\frac{1}{3}\operatorname{arc\,tg}\frac{2\sqrt{\nu\xi}}{\nu-\xi}\right)\right\} \quad (2.29)$$

where

$$2\xi=\frac{2}{3}(-\eta)^{3/2}-\vartheta, \qquad 2\nu=\frac{2}{3}(-\eta)^{3/2}+\vartheta \quad (2.30)$$

On $\nu = 0$ and $\xi = 0$ the arc tg is equal to π and 0, respectively, and on *od* it equals $1/2\ \pi$.

In concluding this paper we note that we have applied the method we have presented to constructing nozzles with a straight transition line. For instance, if the exact solution of the problem of the outflow of gas from a Borda mouthpiece contained in [14] is constructed with our function $d_1\psi_{*2}$, then we obtain a nozzle with a straight transition line in the case of uniform translational flow at infinity upstream.

BIBLIOGRAPHY

1. Pérès, J., *Comptes-rendus Ac. Sci. de Paris*, 20, XI, 219, p. 501, 1944.
2. Pérès, J., Sur l'intégration des équations qui régissent le mouvement d'un fluide compressible. *Proc. Seventh International Congress for Applied Mechanics* Vol. 11, part 11, p. 382-387, 1948.
3. Khristianovich, S.A. and Iur'ev, I.M., Obtekanie profilia pri dokriticheskoi skorosti potoka (The flow about a profile with subcritical flow velocity). *PMM* Vol. 11, No. 1, 1947.
4. Khristianovich, S.A., Obtekanie tel gazom pri bol'shikh dozvukovykh skorostiakh (The flow about bodies by a gas with high subsonic velocities). *Trans. TsAGI*, 481, 1940.
5. Khristianovich, S.A., O sverkhzvukovykh techeniiakh gaza (On supersonic gas flows). *Trans. TsAGI*, 453, 1941.
6. Astrov, V.S., Levin, E., Pavlov, L. and Khristianovich, S.A., O raschete sopel Lavalia (On the calculation of Laval nozzles). *PMM* Vol. 7, No. 17, 1943.
7. Frankl', F.I., K teorii sopel Lavalia (On the theory of Laval nozzles). *Izv. Ak. Nauk SSSR, Ser. mat.* Vol. 9, No. 5, 1945.
8. Fok, V.A., *Tablitsy funktsii Eiri (Tables of Airy functions)*. NII NKEP, 1946.

9. Fal'kovich S.V., K teorii sopla Lavalia (On the theory of a Laval nozzle). *PMM* Vol. 10, No. 3, 1946.

10. Fal'kovich, S.V., Ob odnom klasse sopel Lavalia (On one class of Laval nozzles). *PMM* Vol. 11, No. 2, 1947.

11. Whittaker, E. and Watson, G.N., *Kurs sovremennogo analiza (Course of modern analysis)*. Pt. II, GTTI, 1934.

12. Khristianovich, S.A., Priblizhennoe integrirovanie uravnenii sverkhzvukovogo techeniia gaza (Approximate integration of the equations of supersonic gas flow). *PMM* Vol. 11, No. 2, 1947.

13. Iur'ev, I.M., Priblizhennoe integrirovanie uravnenii ploskogo bezvikhrevogo sverkhzvukovogo dvizheniia gaza (Approximate integration of the equations of plane irrotational supersonic gas motion). *Trans. TsAGI* 1951.

14. Sekerzh-Zen'kovich, Ia.I., K teorii nasadka Borda dlia gaza (On the theory of the Borda mouthpiece for a gas). *PMM* Vol. 21, No. 6, 1957.